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A CONTINUOUS CIRCLE OF PSEUDO-ARCS FILLING UP THE ANNULUS

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To the memory of Professor Bronisław Knaster

Abstract. We prove an early announcement by Knaster on a decomposition of the plane. Then we establish an announcement by Anderson saying that the plane annulus admits a continuous decomposition into pseudo-arcs such that the quotient space is a simple closed curve. This provides a new plane curve, "a selectible circle of pseudo-arcs", and answers some questions of Lewis.

In 1922 the famous construction of an hereditarily indecomposable plane continuum was presented [6] by B. Knaster. Twenty-five years later Moise [11] constructed an hereditarily equivalent and hereditarily indecomposable plane continuum, and called it a pseudo-arc. Further, as a consequence of Bing's [3] characterization of the pseudo-arc, it turned out to be topologically equivalent to the Knaster curve. Moise's result was a starting point to intensive research on this very special continuum by a number of authors (see the survey paper [8]). In this paper we refer to some investigations made by Knaster before Moise's construction. Among the results obtained by Knaster during World War II one can find the following announcement, originally presented in Kiev in 1940:

There exists a real-valued, monotone mapping from the plane that is not constant on any arc.

In the construction Knaster's hereditarily indecomposable continua were exploited. Actually, Knaster's result can be reformulated in the following stronger version (see [10], p. 225):

There exists a real valued, monotone mapping from the plane such that all point-inverses are hereditarily indecomposable.

Unfortunately, Knaster's notes concerning this result were burned during the war, Knaster had never written down the result again, and even his closest ex-students do not know his original idea of construction.

A result similar to that announced by Knaster (with higher dimensional analogues) was proved by Brown [5] in 1958. In fact, a continuous decomposition into hereditarily indecomposable continua of each Euclidean n-space with one point deleted was constructed, such that the real line was the quotient space.

Having no confidence that we follow Knaster's idea, in this paper we construct an example of an open mapping as in the announcement. Then we use it to obtain a continuous decomposition of the plane band (annulus) into pseudo-arcs, such that

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the quotient space is the real line (simple closed curve). This result was announced [1] by Anderson in 1950, but no proof was forthcoming.

In 1959 Bing and Jones [4] constructed a plane circularly chainable curve with a continuous decomposition into pseudo-arcs and with a simple closed curve as the quotient space. They called it a circle of pseudo-arcs and proved its homogeneity. As a subspace of this curve a chainable arc of pseudo-arcs was obtained. Ten years ago Lewis [7] asked whether each curve having a continuous decomposition into pseudo-arcs with an arc as the quotient space is topologically equivalent to that constructed by Bing and Jones. In the present paper we derive some new plane curves from our construction: a new "arc of pseudo-arcs" and a new "circle of pseudo-arcs" (they are triodic and thus topologically different from the curves constructed by Bing and Jones). This answers the question of Lewis in the negative.

Preliminaries

Spaces are metric and mappings are continuous in this paper. The open ε-neighborhood around a set (a point) A is denoted by $N_\varepsilon(A)$.

The symbol dist stands for the usual supremum distance for mappings. Only the uniform convergence is considered for mappings. A mapping $f : X \to Y$ is called an *ε-pushing*, provided X, Y are subspaces of some space U and $d(x, f(x)) < \varepsilon$ for each $x \in X$. The letters $\mathcal{R}$, $\mathcal{Z}$, $\mathcal{C}$ stand for the sets of all real, integer and complex numbers, respectively. The symbols $\overline{u}$,$\overline{v}$,$\overline{w}$, etc., denote vectors. If p is a point of a space X, and X admits translation with vector $\overline{u}$, then $p + \overline{u}$ means the image of p by this translation.

The symbols $\overrightarrow{pq}, \overline{pq}, pq$ denote the vector from p to q, the straight line segment between p and q, and any arc with p and q as its end-points, respectively.

The space of all subcontinua of a space X with the Hausdorff metric is denoted by $C(X)$.

A closed set T in a space X is called *terminal in* X, if either $T \subset K$ or $K \subset T$ for each closed connected set $K \subset X$ with $K \cap T \neq \emptyset$. If each closed connected subset of a connected space X is terminal, then X is said to be *hereditarily indecomposable.* A connected space X is said to be *arc-like* if for any $\varepsilon > 0$ there exists a mapping $f : X \to [0, 1]$ with $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in [0, 1]$. Note that the last two notions are generalizations of the well-known concepts normally used for continua.

Let $\mathcal{D}$ be an upper semi-continuous decomposition of a compact space X. If X admits a retraction $f : X \to Y = f(X) \subset X$ such that $f^{-1}(y) \in \mathcal{D}$ for each $y \in Y$, then Y is called a *continuous selector* of $\mathcal{D}$.

Geometric constructions

In the subsequent investigation we use the following notion. For any $\varepsilon > 0$ we say that a connected space X has *ε-herindpro* (*ε-hereditary indecomposability property*) provided that, for all closed connected sets $K, L \subset X$, if $K \cap L \neq \emptyset$, then either $K \subset N_\varepsilon(L)$, or $L \subset N_\varepsilon(K)$. The following lemma will be used in the main construction.

Lemma 1. *Let a subcontinuum X of a metric space U be the union of two continua X_1, X_2 such that X_1 and X_2 each has ε-herindpro, and there is a δ-pushing $f : X \to pq$ for some arc $pq \subset U$, such that $X_1 \cap X_2 \subset f^{-1}(p)$. Then X has $(\varepsilon + 2\delta)$-herindpro.*

Proof. Let X, X_1, X_2, pq, f be as in the assumptions of the lemma, and let K, L be subcontinua of X with $K \cap L \neq \emptyset$. If $L \subset K$, or $K \subset L$, or $L, K \subset X_1$, or $L, K \subset X_2$, the conclusion is obvious.

If $K \cap f^{-1}(p) \neq \emptyset \neq L \cap f^{-1}(p)$, then either $f(K) \subset f(L)$, or $f(L) \subset f(K)$, and thus $K \subset N_{2\delta}(L)$,or $L \subset N_{2\delta}(K)$.

Assume $K \cap f^{-1}(p) = \emptyset$ and $L \cap f^{-1}(p) \neq \emptyset$. Then either $K \subset X_1 - X_2$, or $K \subset X_2 - X_1$. Assume $K \subset X_1 - X_2$. Further assume $K - N_{(\varepsilon+2\delta)}(L) \neq \emptyset$. Then there is a continuum $L_1 \subset L \cap X_1$ such that $L_1 \cap K \neq \emptyset \neq L_1 \cap f^{-1}(p)$. Since X_1 has ε-herindpro, we have $L_1 \subset N_\varepsilon(K)$. Considering $K \cup L_1$ and L, we have the previous case. Hence $L \subset N_{2\delta}(K \cup L_1) \subset N_{(2\delta+\varepsilon)}(K)$. The other cases are similar. □

Considering the representation of the pseudo-arc as the inverse limit of arcs, in the next lemma we observe the existence of arbitrarily large arcs with ε-herindpro for arbitrarily small ε.

Lemma 2. *For any $M > 0$ and any $\varepsilon > 0$ there are a $\delta > 0$ and a mapping $\lambda : [0, \delta] \to [0, M]$ such that*

(1) *$\lambda(0) = 0$, $\lambda(\delta) = M$, and*

(2) *the graph $\{(x, \lambda(x)) : x \in [0, \delta]\}$ has ε-herindpro.*

Remark. Actually, δ can be arbitrarily chosen from some interval $(0, \delta_0)$.

Proof. To make the lemma evident, it suffices to consider a representation

$$P = \varprojlim(I_n, f_n)$$

of the pseudo-arc P as the inverse limit with $I_n = [0, 1]$, and to take $f_{n,m} = f_m \circ f_{m+1} \circ ... \circ f_n : I_{n+1} \to I_m$ for suitably large m and n. Then the mapping $\lambda(x) = M \cdot f_{n,m}(x/\delta)$ with suitably small δ satisfies the conclusion of the lemma by the hereditary indecomposability of P. The details are left to the reader. □

In our further investigations we consider the space

$$\mathcal{S} = \mathcal{X} \times \mathcal{R},$$

where $\mathcal{R}$ is the set of all reals, and either $\mathcal{X} = \mathcal{R}$, or $\mathcal{X}$ is the circle $S^1 = \{x \in \mathcal{C} : |x| = 1\}$, with the convex metric $d_1(p, q) = \min\{\text{length}\, pq : pq \,\text{is an arc in}\, S^1\}$, for $p, q \in S^1$. If $\mathcal{S} = \mathcal{R}^2$ we consider the usual Euclidean metric in $\mathcal{S}$. If $\mathcal{S} = S^1 \times \mathcal{R}$, then let $d((x, y), (u, w)) = \sqrt{(d_1(x, u))^2 + |y - w|^2}$, for $x, u \in S^1$ and $y, w \in \mathcal{R}$. We consider the usual vector space acting on $\mathcal{S}$. In particular, if $\mathcal{S} = S^1 \times \mathcal{R}$ and $\overline{v} = \langle a, b\rangle$ is a vector with coordinates $a, b \in \mathcal{R}$, then for any $p = (x, y) \in \mathcal{S}$ we have $p + \overline{v} = (x \cdot z, y + b)$, where $z = (\cos a, \sin a)$ and $x \cdot z$ is the usual product of the complex numbers x, z.

We will use the term *stable mapping* on $\mathcal{S}$ in the following stronger meaning, although it differs from that commonly used for the term. A mapping $f : \mathcal{S} \to \mathcal{S}$ is said to be stable provided there is a bounded set $U \subset \mathcal{S}$ such that $f(x) = x$ for each $x \in \mathcal{S} - U$. Two decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathcal{S}$ are said to be *stably equivalent*, if there is a stable homeomorphism $h : \mathcal{S} \to \mathcal{S}$ such that $h(D) \in \mathcal{D}_2$ for each $D \in \mathcal{D}_1$.

Let $f : X \to X$ be a surjection, and $\mathcal{D}_1$, $\mathcal{D}_2$ be decompositions of the space X. We say that f transforms $\mathcal{D}_1$ onto $\mathcal{D}_2$, if $f^{-1}(D) \in \mathcal{D}_1$ for each $D \in \mathcal{D}_2$.

Fix the following decomposition of $\mathcal{S}$:

$$\mathcal{D}_0 = \{\{x\} \times \mathcal{R} : x \in \mathcal{X}\}.$$

The next theorem is crucial in this paper. Applying the theorem, a series of decompositions of the plane or open annulus into lines will be constructed. These will be stably equivalent to the decomposition into vertical lines or radial segments so that each element can be continuously compactified by adding points at the top and bottom, or inside and outside. This sequence of compactified decompositions will converge to the desired decomposition into pseudo-arcs.

Theorem 3. *For each bounded set $U \subset \mathcal{S}$ and for each $M > \operatorname{diam} U$ there are a decomposition $\mathcal{G}$ of $\mathcal{S}$ stably equivalent to $\mathcal{D}_0$ and a stable surjection $g : \mathcal{S} \to \mathcal{S}$ such that*

(1) *g transforms the decomposition $\mathcal{G}$ onto $\mathcal{D}_0$,*
(2) *g is a $(1/M)$-pushing, and*
(3) *for each $G \in \mathcal{G}$ and for each continuum $C \subset U \cap G$ the continuum C has $(1/M)$-herindpro.*

Remark. Observe that the property described in the theorem is invariant with respect to stable homeomorphisms in the sense that, for each homeomorphism $h : \mathcal{S} \to \mathcal{S}$, if a decomposition $\mathcal{D} = \{h(D) : D \in \mathcal{D}_0\}$ has the property, then each decomposition $\mathcal{D}'$ which is stably equivalent to $\mathcal{D}$ has the property, too. In particular, it holds for $\mathcal{D} = \mathcal{D}_0$.

Proof of Theorem 3. The idea of the proof is the following (see Figure 1). First, we construct a homeomorphism $h_1 : \mathcal{S} \to \mathcal{S}$ such that for each straight line $D \in \mathcal{D}_0$ its image lies close to D (in the sense that the Hausdorff distance between them is small enough). In this construction the conclusion of Lemma 2 is employed to assure local $(1/M)$-herindpro for $h_1(D)$. Second, we modify h_1 to a homeomorphism $h : \mathcal{S} \to \mathcal{S}$ satisfying $h(x) = x$ for each x in the complement of a bounded neighborhood of U. Then we obtain the required decomposition $\mathcal{G} = \{h(D) : D \in \mathcal{D}_0\}$. Finally, we take a mapping $g : \mathcal{S} \to \mathcal{S}$ such that for each $D \in \mathcal{D}_0$ the partial mapping $g|h(D)$ is the orthogonal projection of $h(D)$ onto D.

Let $D_1 = \{(x, y) \in S : x = 1\}$ and note that $D_1 \in \mathcal{D}_0$. Fix $\sigma, \varepsilon < 1$ such that $0 < \sigma < \varepsilon/2 < \varepsilon < 1/M$, and let $\overline{u} = \langle 0, 2M\rangle$, $\overline{v} = \langle \sigma, 2M\rangle$, $\overline{v}_0 = (1/|\overline{v}|)\overline{v}$. Define ϕ as the angle between the vectors $\overline{u}$ and $\overline{v}$.

For $\varepsilon/2$ and $2M$ take a δ and a $\lambda : [0, \delta] \to [0, 2M]$ guaranteed by Lemma 2, assuming (see the remark after Lemma 2)

$$\delta < \varepsilon/4 \quad \text{and} \quad \delta < M \sin^2 \phi,$$

and, if $\mathcal{X} = S^1$, we add the following extra assumption on δ :

(3.1) For any point $p \in \mathcal{S}$ and for the spiral line $L(p) = \{p + t\overline{v} : t \in \mathcal{R}\}$, if r_1 and r_2 are two consecutive points of the intersection $L \cap D_1$, then the number $(d(r_1, r_2) \cdot \sin\phi)/2\delta$ is a fixed integer j.

Now we fix two sequences of points $p_k, q_k \in D_1$ defined by

$$p_k = (1, 2k \cdot \delta/\sin\phi) \in \mathcal{S},$$

$$q_k = (1, (2k + 1) \cdot \delta/\sin\phi) \in \mathcal{S}, \text{ for each } k \in \mathcal{Z}.$$

Observe that, if $\mathcal{S} = S^1 \times \mathcal{R}$, by virtue of (3.1) we have

$$L(p_k) \cap D_0 = \{p_{k+mj} : m \in \mathcal{Z}\},$$
$$L(q_k) \cap D_0 = \{q_{k+mj} : m \in \mathcal{Z}\}.$$

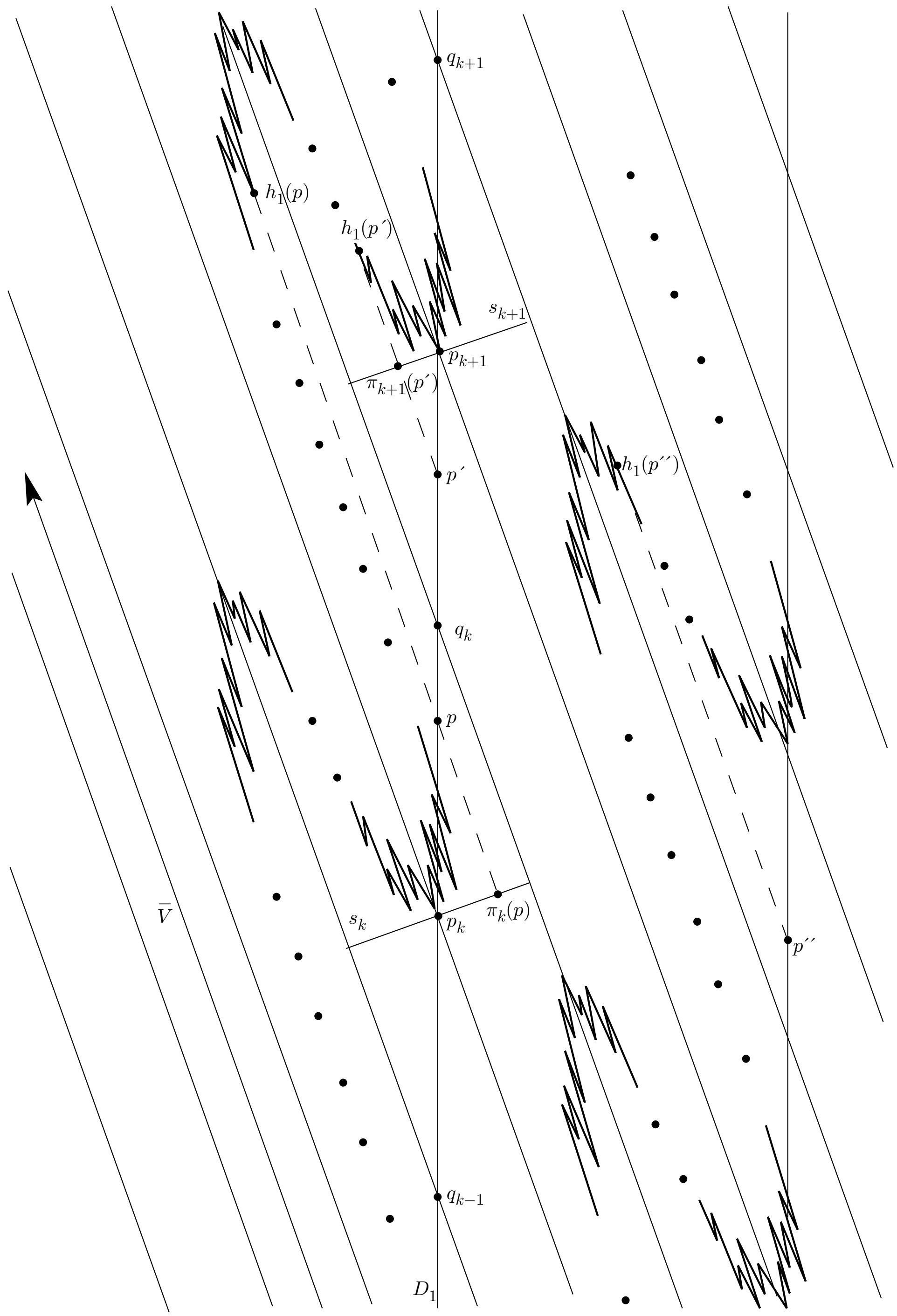

FIGURE 1.

Let s_k denote the segment in $\mathcal{S}$ perpendicular to $\overline{v}$, with length 2δ, having p_k as its midpoint. Let π_k be the orthogonal projection of the segment $\overline{q_{k-1}q_k} \subset D_1$ onto s_k. For any $p \in \overline{p_k q_k}$ let

$$h_1(p) = \pi_k(p) + \lambda(d(p_k, \pi_k(p))) \cdot \overline{v}_0.$$

Next, let $\mu = (M - \delta \cdot \cot\phi)/M$, and observe that

$$1 > \mu = (M - \delta \cdot \cot\phi)/M > (M - M \cdot \sin^2\phi \cdot \cot\phi)/M = 1 - (\sin 2\phi)/2 \geq 1/2.$$

Then, for any $p \in \overline{q_k p_{k+1}}$ let

$$h_1(p) = \pi_{k+1}(p) + \mu\lambda(d(\pi_{k+1}(p), p_{k+1}))\overline{v}_0.$$

Observe that $h_1(p)$ is well defined for all points p in D_1.

Next, note that for a fixed k the set $\{h_1(p) : p \in \overline{p_k q_k}\}$ is isometric to the graph of λ, and the set $\{h_1(p) : p \in \overline{q_k p_{k+1}}\}$ is isometric to the graph of $\mu \cdot \lambda$. Therefore both these sets have ε-herindpro.

Take any continuum K in $h_1(D_1)$ with $\operatorname{diam} K < M$. Since each of $\operatorname{diam} h_1(\overline{p_k q_k})$ and $\operatorname{diam} h_1(\overline{q_k p_{k+1}})$ is greater than M for all k, there is a k such that either $K \subset h_1(\overline{p_k p_{k+1}})$, or $K \subset h_1(\overline{q_k q_{k+1}})$. Assume $K \subset h_1(\overline{p_k p_{k+1}})$ (the other case is similar). The set $h_1(\overline{p_k p_{k+1}}) = h_1(\overline{p_k q_k}) \cup h_1(\overline{q_k p_{k+1}})$ admits a δ-pushing onto the segment $\overline{\pi_k(q_k) h_1(q_k)}$ (the orthogonal projection), and the arcs $h_1(\overline{p_k q_k})$ and $h_1(\overline{q_k p_{k+1}})$ have $(\varepsilon/2)$-herindpro. This implies that $h_1(\overline{p_k p_{k+1}})$ has $(1/M)$-herindpro by Lemma 1 (we have $(\varepsilon/2) + 2\delta < (\varepsilon/2) + 2(\varepsilon/4) = \varepsilon < 1/M$), and thus K also has $(1/M)$-herindpro. Hence all continua in $h_1(D_1)$ with diameters $< M$ have $(1/M)$-herindpro.

To define h_1 for each $p \in \mathcal{S}$ take the projection p' of p into D_1 in the direction of $\overline{v}$, and let

$$h_1(p) = p + \overrightarrow{p'\, h_1(p')}\ .$$

Comment. If $\mathcal{X} = S^1$ the projection is not uniquely determined. However, if p' and p'' are such projections, their distance is a multiple of $2\delta/\sin\phi$ (see (3.1)). Then, from the definition of $h_1(q)$ for $q \in D_1$, it follows that $\overrightarrow{p'\, h_1(p')} = \overrightarrow{p''\, h_1(p'')}$. This implies the unique determination of h_1.

The mapping h_1 is a uniformly continuous homeomorphism of $\mathcal{S}$ onto itself. Moreover, for all $D \in \mathcal{D}_0$ the sets $h_1(D)$ are mutually isometric, and thus for each continuum K contained in any such $h_1(D)$ with $\operatorname{diam} K < M$, the continuum K has $(1/M)$-herindpro.

Now we modify h_1 to make it stable. Take $\alpha > 0$ large enough that $U \subset N_\alpha(p_0)$. Let $\beta = |\overline{v}|$ and observe that $|\overrightarrow{p\, h_1(p)}| \leq \beta$ for each $p \in \mathcal{S}$. Further, define $\gamma : [0, \infty) \to [0, 1]$ by

$$\gamma(x) = \begin{cases} 1 & \text{for} \quad x \in [0, \alpha + \beta], \\ \frac{\alpha + 3\beta - x}{2\beta} & \text{for} \quad x \in [\alpha + \beta, \alpha + 3\beta], \\ 0 & \text{for} \quad x > \alpha + 3\beta. \end{cases}$$

If $\mathcal{X} = \mathcal{R}$, for any $p \in \mathcal{S}$ let

$$h(p) = p + \gamma(d(p, p_0))\ \overrightarrow{p\, h_1(p)},$$

and if $\mathcal{X} = S^1$, for any $p = (x, y) \in \mathcal{S}$ let

$$h(p) = p + \gamma(|x|)\ \overrightarrow{p\, h_1(p)}\ .$$

One can verify that in both cases h is a stable homeomorphism.

Define $\mathcal{G} = \{h(D) : D \in \mathcal{D}_0\}$. Let a continuum $K \subset U$ be contained in an element $h(D)$ of $\mathcal{G}$. Since the mappings h_1, h coincide on $N_{\alpha+\beta}(p_0)$ and $h^{-1}(U) \subset N_{\alpha+\beta}(p_0)$, we have $K \subset h_1(D)$. Further, $\operatorname{diam} K \le \operatorname{diam} U < M$, and thus K has ε-herindpro.

For any $p = (x, y) \in \mathcal{S}$ let $(x_p, y_p) = h^{-1}(p)$. Define $g : \mathcal{S} \to \mathcal{S}$ by

$$g(p) = (x_p, y)\ .$$

Then g is a $(1/M)$-pushing such that for each $D \in \mathcal{D}_0$ we have $g^{-1}(D) = h(D) \in \mathcal{G}$, and g is stable. The proof of Theorem 3 is complete. □

Topological results

In the main construction we employ the following theorem on inverse limits. The idea of this theorem resembles that of a theorem by Anderson and Choquet [2] (Theorem 1, p.348).

Theorem 4. *Let X be a complete metric space and let a sequence $f_n : X \to X$ of surjections f_n satisfy the following two conditions:*

(a) $\sum_{n=1}^{\infty} \operatorname{dist}(f_n, \mathrm{id}_X) < \infty$.

(b) *For each positive integer k the sequence $f_k^{n+k} = f_k \circ f_{k+1} \circ \ldots \circ f_{k+n-1}$ converges uniformly.*

Then the function $h : X \to \varprojlim(X, f_n)$ defined by

$$h(x) = (\lim f_1^{1+n}(x), \lim f_2^{2+n}(x), \ldots)$$

is a homeomorphism, and

$$h^{-1}(x_1, x_2, \ldots) = \lim x_n$$

for each $(x_1, x_2, \ldots) \in \varprojlim(X, f_n)$.

Proof. For any $x \in X$ let $h(x) = (x_1, x_2, \ldots)$. Since

$$\begin{aligned} f_{k-1}(x_k) &= f_{k-1}(\lim_n f_k^{k+n}(x)) = f_{k-1}(\lim_n f_k \circ f_{k-1} \circ \ldots \circ f_{k+n-1}(x)) \\ &= \lim_n f_{k-1} \circ f_k \circ f_{k+1} \circ \ldots \circ f_{k+n-1}(x) = x_{k-1}, \end{aligned}$$

h is well defined.

On the other hand, for any $(y_1, y_2, \ldots) \in \varprojlim(X, f_n)$, observe that y_n converges to some y by condition (a). For any k we have $y_k = f_k^{k+m}(y_{k+m}) \xrightarrow{m} \lim_n f_k^{k+n}(y)$ by condition (b), and thus $(y_1, y_2, \ldots) = h(y)$. Hence h is a one-to-one mapping from X onto $\varprojlim(X, f_n)$. The continuity of h (of h^{-1}) can be proven by condition (b) (by condition (a), respectively). The details are left to the reader. □

The following theorem (in a weaker version; see the introduction) was announced by Knaster in 1940 [10], but the proof has neither been presented, nor published. In this theorem we obtain a continuous decomposition of the plane into “pseudo-lines”, i.e., closed connected sets, each admitting a pseudo-arc as its 2-point compactification. These pseudo-arcs are irreducible between the two points of the compactification.

Theorem 5 (Knaster)**.** *There exists a continuous decomposition $\mathcal{D}$ of the plane $\mathcal{R}^2$ into closed, connected, noncompact, hereditarily indecomposable, arc-like sets, such that the quotient space is homeomorphic to the real line.*

Moreover, for each $\varepsilon > 0$ the decomposition $\mathcal{D}$ can be constructed to admit a surjective ε-pushing $f : \mathcal{R}^2 \to \mathcal{R}^2$ transforming $\mathcal{D}$ onto $\mathcal{D}_0$.

Proof. Fix any $\varepsilon > 0$ and let $U_n = [-n, n]^2 \subset \mathcal{R}^2$. Applying Theorem 3, we construct a proper sequence $\mathcal{D}_0, \mathcal{D}_1, \ldots$ of decompositions of $\mathcal{R}^2$ and a sequence $g_n : \mathcal{R}^2 \to \mathcal{R}^2$ of surjective $(\varepsilon/2^n)$-pushings transforming $\mathcal{D}_n$ onto $\mathcal{D}_{n-1}$, such that for each k the sequence $g_k^{k+n} = g_k \circ g_{k+1} \circ \ldots \circ g_{k+n-1}$ converges as $n \to \infty$. More precisely, the mappings g_k are chosen such that

(5.1) $\operatorname{dist}(g_k^{k+n}, g_k^{k+n+1}) \le \varepsilon/2^{n+k}$ for all k and n.

Then we apply Theorem 4 to obtain the conclusion.

As previously, take $\mathcal{D}_0 = \{\{x\} \times \mathcal{R} : x \in \mathcal{R}\}$, and let $M_0 = \max\{4, 2/\varepsilon\}$. Then take a decomposition $\mathcal{D}_1$ and a mapping $g_1 : \mathcal{R}^2 \to \mathcal{R}^2$ guaranteed by Theorem 3 for $\mathcal{D}_0$, U_1 and M_0.

Assume that we have constructed decompositions $\mathcal{D}_0, \ldots, \mathcal{D}_i$, stable mappings $g_1, \ldots, g_i : \mathcal{R}^2 \to \mathcal{R}^2$, and numbers $M_0, \ldots, M_{i-1}$ such that

(5.2) $\mathcal{D}_j$ is stably equivalent to $\mathcal{D}_0$ for $j \in \{1, \ldots, i\}$,
(5.3) $M_j \ge \max\{4j, 2^{j+1}/\varepsilon\}$ for $j \in \{0, \ldots, i-1\}$,
(5.4) g_j is a $(1/M_{j-1})$-pushing for $j \in \{1, \ldots, i\}$,
(5.5) condition (5.1) is satisfied for $n + k \le i$,
(5.6) g_j transforms $\mathcal{D}_j$ onto $\mathcal{D}_{j-1}$ for $j \in \{1, \ldots, i\}$, and
(5.7) for each $D \in \mathcal{D}_j$ and for each continuum $C \subset D \cap U_j$, the continuum has $(1/M_{j-1})$-herindpro for $j \in \{1, \ldots, i\}$.

Assume $n + k = i + 1$. Let $\delta(n, k) > 0$ be a number such that if $d(p, q) < \delta(n, k)$, then $d(g_k^{k+n}(p), g_k^{k+n}(q)) < \varepsilon/2^{n+k}$. Let $M_i = \max(\{2^{i+1}/\varepsilon\} \cup \{1/\delta(n, k) : k + n = i+1\})$. Then take a decomposition $\mathcal{D}_{i+1}$ and a mapping $g_{i+1} : \mathcal{R}^2 \to \mathcal{R}^2$ guaranteed by Theorem 3 (see also the remark after Theorem 3) for $\mathcal{D}_i$, U_{i+1} and M_i. One can verify that conditions (5.2)-(5.7) are satisfied if i is replaced by $i + 1$.

By the induction we obtain sequences M_n, g_n, $\mathcal{D}_n$ satisfying (5.2)-(5.7) for all i. Applying Theorem 4, observe that $\varprojlim(\mathcal{R}^2, g_n)$ is homeomorphic to $\mathcal{R}^2$ and the mapping $h(x_1, x_2, \ldots) = \lim x_n$ is a homeomorphism between these spaces. Let us identify these spaces by this mapping.

Next, let $f_n : \varprojlim(\mathcal{R}^2, g_n) \to \mathcal{R}^2$ be the projections of this inverse system. Observe that f_n is $(\varepsilon/2^{n-1})$-pushing. Indeed, let $(x_1, x_2, \ldots)$ be a point of the inverse limit identified with $x_0 = \lim x_n$. Then $f_n(x_0) = x_n$, and $d(f_n(x_0), x_0) \le \sum_n^\infty d(x_k, x_{k+1}) < \sum_n^\infty \varepsilon/2^k = \varepsilon/2^{n-1}$. Let

$$\mathcal{D}_\infty = \{f_1^{-1}(D) : D \in \mathcal{D}_0\}$$

and observe that

$$\mathcal{D}_\infty = \{f_n^{-1}(D) : D \in \mathcal{D}_{n-1}\}$$

for each n.

Further, note that $\mathcal{D}_\infty$ is a decomposition of $\mathcal{R}^2$ such that for each $D \in \mathcal{D}_\infty$ there are $D_n \in \mathcal{D}_n$ satisfying $g_n(D_n) = D_{n-1}$ and

$$D = \varprojlim(D_n, g_{n+1}|D_{n+1}).$$

The sets D_n are closed in $\mathcal{R}^2$ and homeomorphic to the real line. Thus D is connected, closed and arc-like.

Let L, K be arbitrary subcontinua of some $D \in \mathcal{D}_\infty$ with $L \cap K \neq \emptyset$. We have $f_n(L) \cup f_n(K) \subset D_{n-1} \cap U_{n-1}$ for sufficiently large n and some $D_{n-1} \in \mathcal{D}_{n-1}$. Then either $f_n(L) \subset N_{\alpha(n)}(f_n(K))$ for infinitely many n, or $f_n(K) \subset N_{\alpha(n)}(f_n(L))$ for infinitely many n, where $\alpha(n) = \varepsilon/2^{n-1}$. Since $\lim f_n = \mathrm{id}_{\mathcal{R}^2}$, it follows that either $L \subset K$, or $K \subset L$.

Assume that L, $K \subset D$ are closed (not necessarily compact), connected sets with $L \cap K \neq \emptyset$. Then there are continua L_n, K_n approximating L, K (respectively) such that $L_n \subset L$, $K_n \subset K$ and $L_n \cap K_n \neq \emptyset$. We have either $L_n \subset K_n$ for infinitely many n, or $K_n \subset L_n$ for infinitely many n. So, again we infer that either $L \subset K$, or $K \subset L$. Hence D is hereditarily indecomposable for each $D \in \mathcal{D}_\infty$.

Finally, each $\mathcal{D}_n$ is continuous and f_n is an $(\varepsilon/2^{n-1})$-pushing. Thus $\mathcal{D}_\infty$ is continuous as well. We take f_1 for the required mapping f. The proof is then complete. □

Remark. Theorem 5 remains true, if we replace $\mathcal{R}^2$ by $S^1 \times \mathcal{R}$. The argument is the same.

Remetrizing $\mathcal{R}^2$ properly, we can identify $\mathcal{R}^2$ with its subset $\mathcal{B}_0 = \{(x, y) \in \mathcal{R}^2 : |y| < 1\}$ such that $\mathcal{D}_0 = \{\{x\} \times (-1, 1) : x \in \mathcal{R}\}$. Define $\mathcal{B} = \{(x, y) \in \mathcal{R}^2 : |y| \leq 1\}$. Then we can extend the mappings $g_n : \mathcal{B}_0 \to \mathcal{B}_0$ used in the proof of Theorem 5 to mappings $g_n^* : \mathcal{B} \to \mathcal{B}$ by letting $g_n^*(x) = x$ for $x \in \mathcal{B} - \mathcal{B}_0$ (the g_n's are stable on $\mathcal{R}^2$). Next, extend the decompositions $\mathcal{D}_n$ of $\mathcal{B}_0$ to decompositions $\mathcal{D}_n^*$ of $\mathcal{B}$ adding the proper pair of points $(x, -1)$, $(x, 1)$ to each element D of $\mathcal{D}_n$ to compactify it to an arc (the $\mathcal{D}_n$'s are stably equivalent to $\mathcal{D}_0$ in $\mathcal{R}^2$). Then, similarly as in the proof of Theorem 5, we obtain a decomposition $\mathcal{D}_\infty^*$ of $\mathcal{B}$. The elements of $\mathcal{D}_\infty^*$ are arc-like and hereditarily indecomposable; thus they are pseudo-arcs [3]. We have obtained the following theorem.

Theorem 6. *There exists a continuous decomposition $\mathcal{D}$ of the plane band $\mathcal{B} = \{(x, y \in \mathcal{R}^2 : |y| \leq 1\}$ into pseudo-arcs, such that:*

(1) *the quotient space is homeomorphic to the real line $\mathcal{R}$, and*
(2) *each pseudo-arc $D \in \mathcal{D}$ intersects the line $y = 1$ at exactly one point p_D, and the line $y = -1$ at exactly one point q_D, and D is irreducible between p_D and q_D. (Each of these lines yields a continuous selection of the decomposition.)*

Moreover, for each $\varepsilon > 0$ the decomposition $\mathcal{D}$ can be constructed to admit a surjective ε-pushing $f : \mathcal{B} \to \mathcal{B}$ transforming $\mathcal{D}$ onto $\mathcal{D}_0^*$.

Applying the modification of Theorem 5 concerning $S^1 \times \mathcal{R}$ (see the remark after Theorem 5), similarly as the above theorem, we obtain the decomposition as in the title of the paper.

Theorem 7. *There exists a decomposition $\mathcal{D}$ of the annulus $\mathcal{A} = \{z \in \mathcal{C} : 1 \leq |z| \leq 2\}$ into pseudo-arcs, such that:*

(1) *the quotient space is a simple closed curve, and*
(2) *each pseudo-arc $D \in \mathcal{D}$ intersects the circle $C_1 = \{z \in \mathcal{C} : |z| = 1\}$ at exactly one point p_D, and the circle $C_2 = \{z \in \mathcal{C} : |z| = 2\}$ at exactly one point q_D, and D is irreducible between p_D and q_D. (Each of these circles yields a continuous selection of $\mathcal{D}$.)*

Moreover, for each $\varepsilon > 0$ the decomposition $\mathcal{D}$ can be constructed to admit a surjective ε-pushing of $\mathcal{A}$ transforming $\mathcal{D}$ onto the decomposition $\mathcal{D}_0^*$ of $\mathcal{A}$ into maximal radial segments.

Let $\mathcal{D}$ be the decomposition as in the above theorem. Without loss of generality assume that the points p_D, q_D are the ends of maximal radial segments in $\mathcal{A}$. Define $f_1 : \mathcal{A} \to C_1$ by $f_1(z) = p_D$ for $z \in D \in \mathcal{D}$. Then f_1 is a monotone, open retraction. Let $\omega : C(\mathcal{A}) \to [0,1]$ be a Whitney map. Let $a_0 = \inf\{\omega(D) : D \in \mathcal{D}\}$ and take any a with $0 < a < a_0$. Observe that since the elements of $\mathcal{D}$ are hereditarily indecomposable, for each $z \in \mathcal{A}$ there is exactly one set $P_z \in \omega^{-1}(a)$ such that $z \in P_z \subset D_z$, where D_z is the element of $\mathcal{D}$ containing z. Thus we obtain a decomposition $\mathcal{P}_a = \{P_z : z \in \mathcal{A}\}$ of $\mathcal{A}$ into pseudo-arcs. We show that this decomposition is continuous.

In fact, note that a decomposition $\mathcal{F}$ of a compact space X into subcontinua of X is continuous if and only if $\mathcal{F}$ is a closed subset of the hyperspace $C(X)$. Observe that the set $\mathcal{G} = \{C \in C(\mathcal{A}) : C \subset D \text{ for some } D \in \mathcal{D}\}$ is closed in $C(\mathcal{A})$ by the continuity of $\mathcal{D}$, and that $\omega^{-1}(a)$ is closed in $C(\mathcal{A})$ by the continuity of ω. Since $\mathcal{P}_a = \mathcal{G} \cap \omega^{-1}(a)$, it follows that $\mathcal{P}_a$ is a closed subset of $C(\mathcal{A})$. So $\mathcal{P}_a$ is continuous.

Further, we prove that the quotient space $\mathcal{A}/\mathcal{P}_a$ is homeomorphic to $\mathcal{A}$. Indeed, extend the decomposition $\mathcal{P}_a$ to $\mathcal{R}^2$ taking singletons outside of $\mathcal{A}$. Then the quotient space is homeomorphic to $\mathcal{R}^2$ by the well-known Moore theorem [12] on upper semi-continuous decompositions of the plane into non-separating plane continua. Finally, we see that $\mathcal{A}/\mathcal{P}_a$ is a subcontinuum of the quotient space with the boundary composed of two disjoint simple closed curves (the images of C_1 and C_2), and thus $\mathcal{A}/\mathcal{P}_a$ is a topological annulus.

Let $g_a : \mathcal{A} \to \mathcal{A}$ be the composition of the quotient map $\mathcal{A} \to \mathcal{A}/\mathcal{P}_a$ and of a homeomorphism $\mathcal{A}/\mathcal{P}_a \to \mathcal{A}$ such that $g_a(z) = z$ for each $z \in C_1 \cup C_2$. Let $U = \{z \in \mathcal{C} : 1 < |z| < 2 \text{ and } 0 < \arg z < \pi\}$. Since g_a is open and monotone, the set $g_a^{-1}(U)$ is an open, connected subset of the plane. Note that $\operatorname{bd} g_a^{-1}(U)$ is the union of four connected sets, namely $\{z \in \mathcal{C} : |z| = j \text{ and } 0 \le \arg z \le \pi\}$ and $g_a^{-1}(\{z \in \mathcal{A} : \arg z = (j-1)\cdot\pi\})$ for $j \in \{1,2\}$, such that the four points $(0,-2)$, $(0,-1)$, $(0,1)$ and $0,2)$ join them to form a connected set. Thus the set $g_a^{-1}(U)$ is simply connected. It follows that $g_a^{-1}(U)$ is homeomorphic to the plane $\mathcal{R}^2$. The decomposition $\mathcal{P}_a$ restricted to $g_a^{-1}(U)$ is a continuous decomposition of $g_a^{-1}(U)$ into pseudo-arcs. Thus we have obtained another proof of the Lewis-Walsh theorem saying that the plane admits a continuous decomposition into pseudo-arcs [9].

Even more interesting are the sets $\tilde{C}_i = g_a^{-1}(C_i)$, $i = 1, 2$. First, note that they are curves. Indeed, each $g_a^{-1}(p)$ has empty interior in $f_1^{-1}(p)$, and we have $f_1^{-1}(p) - g_a^{-1}(p) \subset \mathcal{R}^2 - \tilde{C}_1$. Therefore $\tilde{C}_i$ has empty interior in $\mathcal{R}^2$, and thus it is one-dimensional for $i \in \{1,2\}$. Next, observe that $\mathcal{P}_a$ restricted to $\tilde{C}_1$ is a continuous decomposition of $\tilde{C}_1$ into pseudo-arcs, and the circle C_1 is a continuous selector of this decomposition. Similarly, taking any arc $L \subset C_1$ instead of C_1, we obtain the curve $\tilde{L} = g_a^{-1}(L) \subset \tilde{C}_1$ with the continuous decomposition into pseudo-arcs and with the continuous selector L of this decomposition. Any curve having a continuous decomposition into pseudo-arcs with an arc (a simple closed curve) as a continuous selector of the decomposition we shall call a *selectible arc of pseudo-arcs* (a *selectible circle of pseudo-arcs*), or shortly, an *s-arc of pseudo-arcs* and an *s-circle of pseudo-arcs*. Thus we have obtained the following theorem.

Theorem 8. *There exist a plane s-arc of pseudo-arcs and a plane s-circle of pseudo-arcs.*

In [8] Lewis asked whether curves admitting continuous decomposition into pseudo-arcs with an arc as the quotient space are topologically unique (Question 1), and more specifically, whether the elements of such a decomposition must be terminal (Question 2). The above examples show that the answer to both these question is no. The known arcs of pseudo-arcs (circles of pseudo-arcs) are chainable (circularly chainable), while the examples constructed here are triodic, and thus topologically different. Evidently, their elements of the decompositions are not terminal.

One can verify that the constructed s-arcs of pseudo-arcs do not separate the plane, and thus they are tree-like. We observe that this is a property of each s-arc of pseudo-arcs.

Proposition 9. *Each s-arc of pseudo-arcs is a tree-like continuum.*

Indeed, this is a consequence of a theorem of Sher on upper semi-continuous decompositions of compacta with elements of trivial shape ([13], Theorem 11).

Corollary 10. *Each s-arc of pseudo-arcs has only one selector.*

In view of the above conclusions the following question seems to be interesting.

Question 1. *Are all s-arcs of pseudo-arcs topologically equivalent?*

It is easy to see that for each graph G we can construct a curve $\tilde{G}$ with continuous decomposition into pseudo-arcs having a continuous selector homeomorphic to G. Indeed, we represent G as the union of a finite family of arcs $p_i q_i$ that intersect only at end-points. Then we take a corresponding family of mutually disjoint s-arcs of pseudo-arcs with selectors $p_i' q_i'$, identify corresponding end-points of the selectors, and extend this identification to all the end elements of the decompositions. Now, following the idea of Lewis from [7], we can prove the next theorem.

Theorem 11. *For each curve M there exists a curve $\tilde{M}$ having a continuous decomposition into pseudo-arcs with a continuous selector M' of this decomposition, homeomorphic to M.*

Proof. We represent M as $\varprojlim(G_n, f_n)$, where each G_n is a graph and each f_n is a piecewise linear, finite-to-one mapping. Then we inductively construct continuous curves $\tilde{G}_n$ of pseudo-arcs having G_n's as the continuous selectors with the natural mapping $q_n : \tilde{G}_n \to G_n \subset \tilde{G}_n$ being open and monotone retractions, and extensions $\tilde{f}_n : \tilde{G}_n \to \tilde{G}_{n-1}$ so that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{G}_{n-1} & \xleftarrow{\tilde{f}_n} & \tilde{G}_n \\
\downarrow q_{n-1} & & \downarrow q_n \\
G_{n-1} & \xleftarrow{f_n} & G_n
\end{array}
$$

Indeed, we construct $\tilde{G}_1$ and the retraction $q_1 : \tilde{G}_1 \to G_1$ as described just before the formulation of the theorem.

Assume that for some $n \geq 1$ the space $\tilde{G}_n$ and the mapping $q_n : \tilde{G}_n \to G_n$ is defined. Consider $f_{n+1} : G_{n+1} \to G_n$, where G_{n+1} and G_n are the unions of finitely many free arcs

$$G_n = a_1b_1 \cup \ ... \ \cup a_kb_k \ , \tag{1}$$

$$G_{n+1} = c_1d_1 \cup \ ... \ \cup c_md_m \ , \tag{2}$$

such that each pair of arcs $a_\alpha b_\alpha,\ a_\beta b_\beta$ and $c_\gamma d_\gamma,\ c_\delta d_\delta$ has at most one end point in common (for $\alpha \neq \beta$ and $\gamma \neq \delta$), and there is a function

$$\phi : \{1, ..., m\} \to \{1, ..., k\}$$

such that the mapping

$$f_{n+1}|c_\gamma d_\gamma : c_\gamma d_\gamma \to a_{\phi(\gamma)}b_{\phi(\gamma)}$$

is a homeomorphism. To each arc $a_\alpha b_\alpha \subset G_n$ we define the s-arc of pseudo-arcs $P_\alpha = q_n^{-1}(a_\alpha b_\alpha)$. Further, to each arc $c_\gamma d_\gamma \subset G_{n+1}$ in the union (2), we attach an s-arc of pseudo-arcs Q_γ such that $c_\gamma d_\gamma \subset Q_\gamma$ is the selector of Q_γ and $Q_\gamma \cap Q_\delta = c_\gamma d_\gamma \cap c_\delta d_\delta$ for $\gamma \neq \delta$ in such a manner that there is a homeomorphism $h_\gamma : Q_\gamma \to P_{\phi(\gamma)}$ satisfying $h_\gamma|c_\gamma d_\gamma = f_{n+1}|c_\gamma d_\gamma$. Let $Y = \bigcup\{Q_\gamma : \gamma \in \{1, ..., m\}\}$.

Define two auxiliary mappings on Y: the first, $g_{n+1} : Y \to G_{n+1}$, is the natural projection (being a monotone retraction), and the second, $f'_{n+1} : Y \to \tilde{G}_n = \bigcup\{P_\alpha : \alpha \in \{1, ..., k\}\}$, is defined by

$$f'_{n+1}(y) = h_\gamma(y) \ \text{ for } \ y \in Q_\gamma.$$

Consider an equivalence relation $\star$ on Y for each $y_1,\, y_2 \in Y$, putting $y_1 \star y_2$ provided that

$$g_{n+1}(y_1) = g_{n+1}(y_2) \ \text{ and } \ f'_{n+1}(y_1) = f'_{n+1}(y_2).$$

Then the quotient space $Y/\star$ is just $\tilde{G}_{n+1}$, and the mappings $\tilde{f}_{n+1} : \tilde{G}_{n+1} \to \tilde{G}_n$ and $q_{n+1} : \tilde{G}_{n+1} \to G_{n+1}$ are defined in a natural way so that the diagram

$$\begin{tikzcd}
 & & Y \arrow[dll, "f'_{n+1}"'] \arrow[drr, "g_{n+1}"] & & \\
\tilde{G}_n \arrow[drr, "q_n"'] & & \tilde{G}_{n+1} = Y/\star \arrow[ll, "\tilde{f}_{n+1}"'] \arrow[rr, "q_{n+1}"] & & G_{n+1} \arrow[dll, "f_{n+1}"] \\
 & & G_n & &
\end{tikzcd}$$

commutes. The reader can verify in a routine way that $\tilde{G}_{n+1}$ and $\tilde{f}_{n+1}$ have the required properties.

We define $\tilde{M} = \varprojlim(\tilde{G}_n, \tilde{f}_n)$ and we observe that $M = \varprojlim(G_n, f_n)$ is contained in $\tilde{M}$. Since each $q_n : \tilde{G}_n \to G_n$ is a monotone, open retraction, so is the mapping $q = \varprojlim q_n : \tilde{M} \to M$. Denote by $\pi_n : \tilde{M} \to \tilde{G}_n$ the projection choosing the n-th coordinate. Note that for any $p \in M$ the sets $P_n(p) = q_n^{-1}(\pi_n(p))$ are pseudo-arcs, while the mappings $\tilde{f}_n|P_{n+1}(p) : P_{n+1}(p) \to P_n(p)$ are homeomorphisms. This implies that the set $q^{-1}(p) = \varprojlim(P_n(p), \tilde{f}_n|P_{n+1}(p))$ is a pseudo-arc. The proof is complete. $\square$

A curve C will be called an *s-curve of pseudo-arcs* (a *selectible curve of pseudo-arcs*), provided C admits a continuous decomposition into pseudo-arcs, and C contains a continuous selector of this decomposition. Thus the continua $\tilde{M}$ guaranteed by Theorem 11 are s-curves of pseudo-arcs. Similarly as in Proposition 9 we obtain the following.

Proposition 12. *If X is an s-curve of pseudo-arcs with a continuous selector S, then X is a tree-like continuum if and only if S is tree-like.*

Proposition 13. *If X is a tree-like s-curve of pseudo-arcs, then its continuous selector is uniquely determined. Moreover, for such X, its continuous decomposition into pseudo-arcs admitting a continuous selection is unique.*

Proof. Suppose S_1 and S_2 are continuous selectors of a tree-like s-curve of pseudo-arcs X with some point $x \in S_1 - S_2$. Take a continuum $K \subset S_1 - S_2$ containing x and some point $y \neq x$. Then there are pseudo-arcs P_x, P_y containing x and y, respectively, such that $(P_x - \{x\}) \cup (P_y - \{y\}) \subset (X - S_1) - S_2$. Thus the continuum $L = P_x \cup K \cup P_y \subset X - S_2$ is decomposable. Therefore it must intersect two different pseudo-arcs P_1, P_2 from the decomposition associated with the selector S_2. Observe that the continuum $L \cup P_1 \cup S_2 \cup P_2 \subset X$ is not unicoherent, an impossibility.

Let S be the unique continuous selector of X. Suppose there are two different continuous decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ of X into pseudo-arcs such that S is the continuous selector of each of them. Then there are a point p in $X - S$, two different points x and y in S, and two pseudo-arcs $P_1 \in \mathcal{D}_1$ and $P_2 \in \mathcal{D}_2$ such that $x \in P_1$, $y \in P_2$ and $p \in P_1 \cap P_2$. This implies that $(P_1 \cup P_2) \cap S = \{x, y\}$, and consequently the union $S \cup P_1 \cup P_2 \subset X$ is not unicoherent, an impossibility by the tree-likeness of X. The proof is complete. □

The following question naturally arises.

Question 2. *Must an arbitrary s-curve of pseudo-arcs have only one continuous selector?*

Now we investigate the topological properties of the selectors of plane s-curves of pseudo-arcs. Suppose an arc pq is contained in the continuous selector S of a plane s-curve of pseudo-arcs X with the quotient mapping $g : X \to S$. Let $\{x_n\}$ be a sequence in pq converging to some $x_0 \in pq - \{p, q\}$. Take an open ball $N_\varepsilon(x_0)$ such that $p, q \notin N_\varepsilon(x_0)$. Then the component C of $pq \cap N_\varepsilon(x_0)$ containing x_0 separates $N_\varepsilon(x_0)$ into exactly two components, U_1 and U_2. Further, since $g^{-1}(x_0)$ is acyclic, all sufficiently small subcontinua of $g^{-1}(x_0)$ containing x_0 access x_0 from exactly one of these U_i's (say from U_1). Observe that for almost all n, all sufficiently small subcontinua of $g^{-1}(x_n)$ lie in $U_1 \cup \{x_n\}$ by the continuity of the decomposition of X. Note that this implies the following proposition.

Proposition 14. *No selector of any planar s-curve of pseudo-arcs contains a simple triod.*

Now we prove the next proposition.

Proposition 15. *Each selector of any planar s-curve of pseudo-arcs is hereditarily locally connected.*

Indeed, let X be a plane s-curve of pseudo-arcs with the selector S and the quotient mapping $g : X \to S$. Suppose, on the contrary, that S contains a nondegenerate continuum K_0 and subcontinua K_n converging to K_0 with $K_i \cap K_j = \emptyset$

for all $i, j = 0, 1, \ldots$. Take three different points a, b, $c \in K_0$ and sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ converging to a, b, c, respectively, such that a_n, b_n, $c_n \in K_n$. Then the triods $K_n \cup g^{-1}(a_n) \cup g^{-1}(b_n) \cup g^{-1}(c_n)$ must intersect the triod $K_0 \cup g^{-1}(a_0) \cup g^{-1}(b_0) \cup g^{-1}(c_0)$ for almost all n, an impossibility. $\square$

Propositions 14 and 15 imply that the selectors of planar s-curves of pseudo-arcs are either arcs, or simple closed curves. We have obtained the following theorem.

Theorem 16. *Each planar s-curve of pseudo-arcs is either an s-arc of pseudo-arcs, or an s-circle of pseudo-arcs.*

As in the above construction of the plane s-arc of pseudo-arcs $g_a^{-1}(L)$ (taking different a from $(0, a_0]$), for any s-arc of pseudo-arcs X we obtain a continuous, monotone family X_t of s-arcs of pseudo-arcs, $t \in (0, T]$, with $X = X_T$ and some X_0 as the common selector for all X_t, satisfying $X_t \subset X_u$ if and only if $t \leq u$.

Question 3. *If X_1, X_2 are (planar) s-arcs of pseudo-arcs with the common continuous selector X_0 and with $X_1 \subset X_2$, does it follow that there is a continuous mapping (a retraction) $f : X_2 \to X_1$ with $f(x) = x$ for $x \in X_0$?*

Finally, we ask two questions on self-homeomorphisms of s-arcs of pseudo-arcs. Assume that X is an arbitrary (planar) s-arc of pseudo-arcs with the continuous selector X_0 and the quotient mapping $q : X \to X_0$.

Question 4. *Given a homeomorphism $h : X_0 \to X_0$, can h be extended to a homeomorphism $h^* : X \to X$?*

Question 5. *Given a point $p \in X_0$ and a homeomorphism $h : q^{-1}(p) \to q^{-1}(p)$ with $h(p) = p$, can h be extended to a homeomorphism $h^* : X \to X$?*

Now we will present some further possible constructions that can be derived from Theorem 11. Namely, we construct some other curves with continuous decomposition into nondegenerate subcontinua having a continuous selector. In particular, we will see that the elements of such decompositions need not be indecomposable.

First, we take a pseudo-arc P_1 for the continuum M in Theorem 11. Then we obtain a selectible pseudo-arc of pseudo-arcs $P_2 = \tilde{M}$. Observe that P_2 is a decomposable, tree-like curve (Proposition 12) that is not embeddable into the plane (Theorem 16 and Proposition 13). Similarly, we apply Theorem 11 to P_2, obtaining thus a tree-like curve P_3 with P_2 as a continuous selector of its continuous decomposition into pseudo-arcs. Observe that P_3 admits also a continuous decomposition into continua each having properties similar to those of P_2 (we do not know if they are topologically unique), with P_1 as the quotient space. Continuing this procedure by transfinite induction (for limit ordinals we take inverse limits with the quotient mappings), for any countable ordinal $\alpha > 0$ we obtain a metric, tree-like curve P_α such that $P_{\alpha+1}$ admits a continuous decomposition into pseudo-arcs with P_α as a continuous selector. We do not know whether the sequence $\{P_\alpha\}$ is topologically unique. However, one can prove that for a fixed such sequence, continua P_α and P_β are nonhomeomorphic if $\alpha \neq \beta$.

Now we repeat the above construction taking any continuum M_0 with dimension ≤ 1 instead of P_1, obtaining thus a transfinite sequence $\{M_\alpha\}$ for countable ordinals $\alpha \geq 0$. One can verify that for each pair of countable ordinals $\alpha \geq 0$ and $\beta > 0$ the curve $M_{\alpha+\beta}$ admits a continuous decomposition into continua of type P_β from the previous sequence, with M_α as the quotient space.

I end the paper with the following two announcements. In a subsequent paper I will prove them, applying the constructions introduced in the present one.

Announcement 1. *Each compact 2-manifold M (with or without boundary) admits a continuous decomposition into pseudo-arcs with the quotient space homeomorphic to M.*

Announcement 2. *The Sierpiński universal plane curve S admits a continuous decomposition into pseudo-arcs with the quotient space homeomorphic to S.*

Added in proof. The proofs of the above announcements have already been published in the paper: J. R. Prajs, *Continuous decompositions of Peano plane continua into pseudo-arcs*, Fund. Math. **158** (1998), 23–40.

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